

Invariants of the adjoint action in the nilradical of a parabolic subalgebra of types B_n, C_n, D_n

V. Sevostyanova*

ABSTRACT. We study invariants of the adjoint action of the unipotent group in the nilradical of a parabolic subalgebra of types B_n, C_n, D_n . We introduce the notion of expanded base in the set of positive roots and construct an invariant for every root of the expanded base. We prove that these invariants are algebraically independent. We also give an estimate of the transcendence degree of the invariant field.

Let K be an algebraically closed field of characteristic zero. For a fixed positive integer n , let G denote one of the following classical groups defined over K : the symplectic group $\mathrm{Sp}_{2n}(K)$, the even orthogonal group $\mathrm{O}_{2n}(K)$, and the odd orthogonal group $\mathrm{O}_{2n+1}(K)$. Throughout the paper, we set

$$m = \begin{cases} 2n & \text{if either } G = \mathrm{Sp}_{2n}(K) \text{ or } \mathrm{O}_{2n}(K); \\ 2n + 1 & \text{if } G = \mathrm{O}_{2n+1}(K). \end{cases}$$

By $\mathrm{U}_m(K)$ denote the upper unitriangular group consisting of all uppertriangular matrices with unit elements on the diagonal. Let $\mathrm{B}_m(K)$ be the Borel group consisting of all upper triangular matrices with nonzero elements on the diagonal. Denote

$$N = G \cap \mathrm{U}_m(K) \text{ and } B = G \cap \mathrm{B}_m(K).$$

Let $P \supset B$ be a parabolic subgroup of the group G . Denote by $\mathfrak{p}, \mathfrak{b}, \mathfrak{n}$ the Lie subalgebras in the Lie algebra $\mathfrak{g} = \mathrm{Lie} G$ that correspond to the Lie subgroups P, B , and N . We represent $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ as the direct sum of

*This research was partially supported by the RFBR (project 08-01-00151-a)

the nilradical \mathfrak{m} and the block diagonal subalgebra \mathfrak{r} with size of blocks (r_1, \dots, r_s) . Consider the adjoint action of the group P in the nilradical \mathfrak{m} :

$$\mathrm{Ad}_g x = gxg^{-1}, \quad x \in \mathfrak{m}, \quad g \in P.$$

The subalgebra \mathfrak{m} is invariant relative to the adjoint action of the group P . We extend this action to the regular representation in the algebra $K[\mathfrak{m}]$ and in the field $K(\mathfrak{m})$:

$$\mathrm{Ad}_g f(x) = f(g^{-1}xg), \quad f(x) \in K(\mathfrak{m}), \quad g \in P.$$

Since the subalgebra \mathfrak{m} is invariant relative to the adjoint action of the group P , it follows that \mathfrak{m} is invariant relative to the action of the Lie subgroup N . The question concerning the structure of the algebra of invariants $K[\mathfrak{m}]^N$ is open and seems to be a considerable challenge (see [6]). We construct a system of invariants

$$\{M_\xi, \xi \in S, L_\varphi, \varphi \in \Phi\}$$

(see Notation 11 and (1)) in the present paper. We show that this system of invariants is algebraically independent over K . We also get an estimate of the transcendence degree of the invariant field:

$$\mathrm{trdeg} K(\mathfrak{m})^N \geq |S| + |\Phi|.$$

We show in paper [5] that the estimate for the case of type A_n is sharp. This question is open for the other types.

Let T be the maximal torus of G consisting of all diagonal matrices and $\Delta = \Delta(G, T)$ be the root system defined by T (see [2]). By definition, Δ is a subset of the Abelian group $X(T) = \mathrm{Hom}(T, K)$ consisting of homomorphisms from T to K . Let $1 \leq i \leq n$. Denote by ε_i an element of the group $X(T)$ such that $\varepsilon_i(t) = t_i$ for all $t \in T$. Here we denote by $t_i \in K$ the (i, i) th entry of the matrix $t \in T$. Then $\Delta = \Delta^+ \cup \Delta^-$ and

$$\Delta^+ = \begin{cases} \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{2\varepsilon_i : 1 \leq i \leq n\} & \text{if } G = \mathrm{Sp}_{2n}(K); \\ \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} & \text{if } G = \mathrm{O}_{2n}(K); \\ \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\varepsilon_i : 1 \leq i \leq n\} & \text{if } G = \mathrm{O}_{2n+1}(K). \end{cases}$$

The roots in Δ^+ are said to be *positive* (and the roots in Δ^- are said to be *negative*). The system of positive roots $\Delta^+_{\mathfrak{r}}$ of the reductive subalgebra \mathfrak{r} is a subsystem in Δ^+ .

We consider the mirror order \prec on the set $\{0, \pm 1, \dots, \pm n\}$, which is defined as

$$1 \prec 2 \prec \dots \prec n \prec 0 \prec -n \prec \dots \prec -2 \prec -1.$$

We index the rows (from left to right) and columns (from top to bottom) of any $m \times m$ matrix according to this order.

For any $\gamma \in \Delta^+$, we set

$$\mathcal{E}(\gamma) = \begin{cases} (-j, -i) & \text{if } \gamma = \varepsilon_i - \varepsilon_j, \ 1 \leq i < j \leq n; \\ (j, -i) & \text{if } \gamma = \varepsilon_i + \varepsilon_j, \ 1 \leq i < j \leq n; \\ (i, -i) & \text{if } G = \mathrm{Sp}_{2n}(K) \text{ and } \gamma = 2\varepsilon_i, \ 1 \leq i \leq n; \\ (0, -i) & \text{if } G = \mathrm{O}_{2n+1}(K) \text{ and } \gamma = \varepsilon_i, \ 1 \leq i \leq n. \end{cases}$$

We define a relation in Δ^+ for which

$$\gamma' \succ \gamma \text{ whenever } \gamma' - \gamma \in \Delta^+.$$

If $\gamma' \succ \gamma$ or $\gamma' \prec \gamma$, then the roots γ and γ' are said to be *comparable*.

Denote by M the set of roots $\gamma \in \Delta^+$ such that the corresponding subalgebras \mathfrak{g}_γ are in \mathfrak{m} . We identify the algebra $K[\mathfrak{m}]$ with the polynomial algebra in the variables $x_{i,j}$, $i \prec j$, whenever $(-j, -i) = \mathcal{E}(\gamma)$ and $\gamma \in M$.

Definition 1. A subset S in M is called a *base* if the elements in S are not pairwise comparable and for any $\gamma \in M \setminus S$ there exist $\xi \in S$ such that $\gamma \succ \xi$.

Definition 2. Let A be a subset in S . We say that γ is a *minimal element* in A if there is no $\xi \in A$ such that $\gamma \succ \xi$.

Note that M has a unique base S , which can be constructed in the following way. We form a set S_1 of minimal elements in M . By definition, $S_1 \subset S$. Then we form a set M_1 , which is obtained from M by deleting S_1 and all elements

$$\{\gamma \in M : \exists \xi \in S_1, \gamma \succ \xi\}.$$

The subset of minimal elements S_2 in M_1 is also contained in S , and so on. Continuing this process, we get the base S as a union of the sets S_1, S_2, \dots

Lemma 3. Let $\gamma \in M \setminus S$.

1. Suppose that $\gamma = \varepsilon_i + \varepsilon_j$, $i < j$, is such that $\mathcal{E}(\xi)$ equals one of the following values: $(j, -k)$, where $-i \succ -k$, $(k, -i)$, where $j \prec k$, or $(k, -j)$ for some k .
2. If $\gamma = \varepsilon_i - \varepsilon_j$, $i < j$, then there exists a root $\xi \in S$ such that $\mathcal{E}(\xi)$ equals $(-j, k)$, where $-i \succ k$, or $(k, -i)$, where $j \prec k$.
3. Let $\gamma = 2\varepsilon_i$, then there exists $\xi \in S$ such that $\mathcal{E}(\xi) = (k, -i)$, where $k \succ i$.

PROOF. We prove the first item of the lemma. The other items are proved similarly.

Let $\gamma = \varepsilon_i + \varepsilon_j \in M \setminus S$, $i < j$. By Definition 1, there exists a root $\xi \in S$ such that $\gamma - \xi \in \Delta^+$. We write all suitable roots ξ .

1. $\xi = \varepsilon_k + \varepsilon_j$, where $i < k$; hence if $k < j$, then $\mathcal{E}(\xi) = (j, -k)$, if $j < k$, then $\mathcal{E}(\xi) = (k, -j)$.
2. $\xi = \varepsilon_i + \varepsilon_k$, where $k > j$; then $\mathcal{E}(\xi) = (k, -i)$.
3. $\xi = \varepsilon_i - \varepsilon_k$, where $i < k$; then $\mathcal{E}(\xi) = (-k, -i)$.
4. $\xi = \varepsilon_j - \varepsilon_k$, where $j < k$; then $\mathcal{E}(\xi) = (-k, -j)$.
5. $\xi = \varepsilon_i$; then $\mathcal{E}(\xi) = (0, -i)$ if $G = \mathrm{O}_{2n+1}(K)$.
6. $\xi = \varepsilon_j$; then $\mathcal{E}(\xi) = (0, -j)$ if $G = \mathrm{O}_{2n+1}(K)$.
7. $\xi = 2\varepsilon_i$; then $\mathcal{E}(\xi) = (i, -i)$ if $G = \mathrm{Sp}_{2n}(K)$.
8. $\xi = 2\varepsilon_j$; then $\mathcal{E}(\xi) = (j, -j)$ if $G = \mathrm{Sp}_{2n}(K)$.

Thus we have proved the lemma. \square

Corollary. *Let $G = \mathrm{Sp}_{2n}(K)$, and let a number $i > 0$ be such that there exists a root $\gamma \in M$, where $\mathcal{E}(\gamma) = (j, -i)$ for some j . Then there exists a root $\xi \in S$ such that $\mathcal{E}(\xi) = (k, -i)$ for some k .*

PROOF. Suppose that there is no root $\xi \in S$ such that $\mathcal{E}(\gamma) = (j, -i)$. Consider a root $2\varepsilon_i \in M$. By Lemma 4, there exists $\xi \in S$ such that $\mathcal{E}(\xi) = (k, -i)$ for some k , which contradicts the assumption. \square

Definition 4. An ordered collection of positive roots $\{\gamma_1, \dots, \gamma_s\}$ is called a *chain* if $\mathcal{E}(\gamma_1) = (a_1, a_2)$, $\mathcal{E}(\gamma_2) = (a_2, a_3)$, $\mathcal{E}(\gamma_3) = (a_3, a_4)$, and so on.

Let $(r_1, \dots, r_{p-1}, r_p, r_{p-1}, \dots, r_1)$ be the sizes of the blocks in the reductive subalgebra \mathfrak{r} . We denote $R = \sum_{t=1}^{p-1} r_t$. Let $\gamma \in S$ be a root such that $\mathcal{E}(\gamma) = (a, -b)$, $b > 0$. If $R \prec a \prec -R$, then γ is called a root lying *to the right of the central block* in the subalgebra \mathfrak{r} .

Let k be the greatest number such that the root $\gamma = \varepsilon_k + \varepsilon_{k+1}$ lies in the root system M . Denote

$$\Gamma_{\mathfrak{r}} = \begin{cases} \Delta_{\mathfrak{r}}^+ \cup \{2\varepsilon_i : k < i \leq n\} & \text{if } G = \mathrm{O}_m(K), \mathrm{O}_{2n}(K); \\ \Delta_{\mathfrak{r}}^+ \setminus \{2\varepsilon_i : k < i \leq n\} & \text{if } G = \mathrm{Sp}_m(K). \end{cases}$$

Definition 5. Assume that one of two roots $\xi, \xi' \in S$ does not lie to the right of the central block in \mathfrak{r} . We say that two roots ξ, ξ' form an *admissible pair* $q = (\xi, \xi')$ if there exists a root α_q in the set $\Delta_{\mathfrak{r}}^+$ such that the collection of roots $\{\xi, \alpha_q, \xi'\}$ is a chain.

Suppose that two roots $\xi, \xi' \in S$ are to the right of the central block in \mathfrak{r} and $\mathcal{E}(\xi) = (a, -b)$, $\mathcal{E}(\xi') = (a', -b')$. Similarly, a pair $q = (\xi, \xi')$ is called an *admissible pair* if there exists a root $\alpha_q \in \Gamma_{\mathfrak{r}}$ such that $\mathcal{E}(\alpha_q) = (-a, a')$.

Note that the root α_q can be found from q uniquely.

We form the set $Q = Q(\mathfrak{p})$ that consists of admissible pairs of roots in S . Let roots ξ and ξ' form an admissible pair. Assume that the roots ξ and ξ' are such that $\mathcal{E}(\xi) = (a, -b)$, $\mathcal{E}(\xi') = (a', -b')$, and $a \preccurlyeq a'$. For every admissible pair $q = (\xi, \xi')$ we construct a positive root $\varphi_q = \alpha_q + \xi'$. Consider the subset $\Phi = \{\varphi_q : q \in Q\}$.

Definition 6. The set $S \cup \Phi$ is called an *expanded base*.

Example 7. In the Lie algebra $\mathfrak{g} = \mathfrak{o}_{16}(K)$, let the reductive subalgebra \mathfrak{r} have the following sizes of diagonal blocks: $(3, 1, 2, 4, 2, 1, 3)$. Now we write all roots in the expanded base:

$$\begin{aligned} S = \{ & \xi_1 = \varepsilon_6 - \varepsilon_7, \quad \xi_2 = \varepsilon_5 - \varepsilon_8, \quad \xi_3 = \varepsilon_4 - \varepsilon_5, \\ & \xi_4 = \varepsilon_3 - \varepsilon_4, \quad \xi_5 = \varepsilon_2 - \varepsilon_6, \quad \xi_6 = \varepsilon_1 + \varepsilon_8 \}. \end{aligned}$$

We write the set of admissible pairs and the corresponding roots of the system Φ .

$$\begin{aligned} Q = \{ & (\xi_1, \xi_1), (\xi_1, \xi_2), (\xi_1, \xi_6), (\xi_2, \xi_2), (\xi_1, \xi_3) \}, \\ \Phi = \{ & \varphi_{(\xi_1, \xi_1)} = \varepsilon_6 + \varepsilon_7, \quad \varphi_{(\xi_1, \xi_2)} = \varepsilon_6 + \varepsilon_8, \quad \varphi_{(\xi_1, \xi_6)} = \varepsilon_6 - \varepsilon_8, \\ & \varphi_{(\xi_2, \xi_2)} = \varepsilon_5 + \varepsilon_8, \quad \varphi_{(\xi_1, \xi_3)} = \varepsilon_4 - \varepsilon_6 \}. \end{aligned}$$

From the given parabolic subalgebra, we construct a square diagram and mark in the diagram each root of the expanded base. Suppose that the positive root γ corresponds to the pair of integers $\mathcal{E}(\gamma) = (j, -i)$, $i > 0$. Thus we mark the root γ in $(j, -i)$ entry of the diagram. We label a root of the set S by the symbol \otimes and a root of Φ by the symbol \times . The other entries in the diagram are empty.

Example 8. Let $G = \mathrm{O}_{16}(K)$. Let the sizes of diagonal blocks of the reductive subalgebra be as in Example 7. Then we have the following diagram:

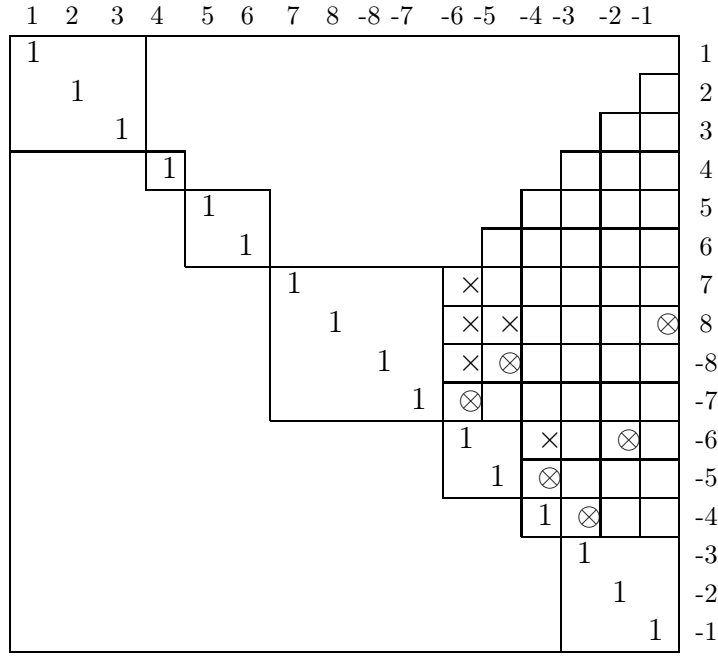


Diagram 1

Example 9. Let $G = \mathrm{Sp}_{16}(K)$. Let the sizes of diagonal blocks of the reductive subalgebra be as in Example 8. Then we get the following diagram.

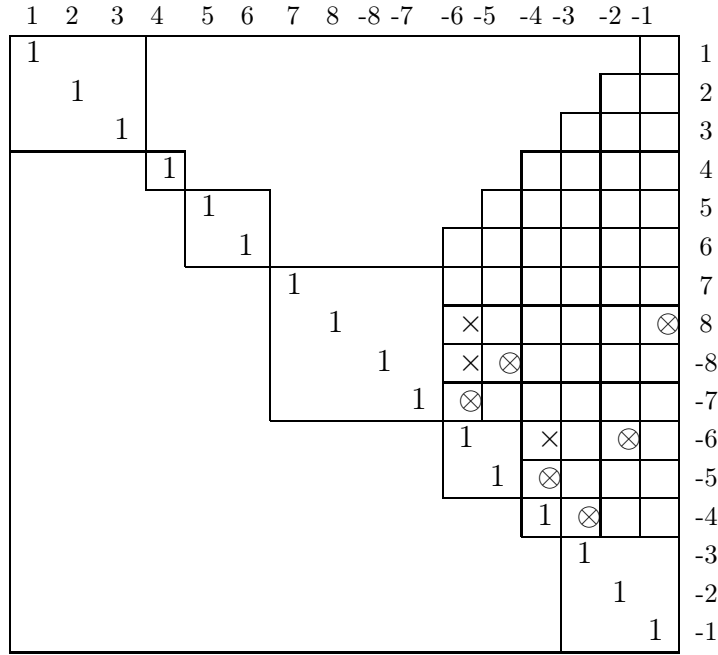


Diagram 2

Пример 10. Let $G = O_{17}(K)$. Let $(1, 2, 5, 1, 5, 2, 1)$ be the sizes of diagonal blocks in the reductive subalgebra. Then we have the following diagram.

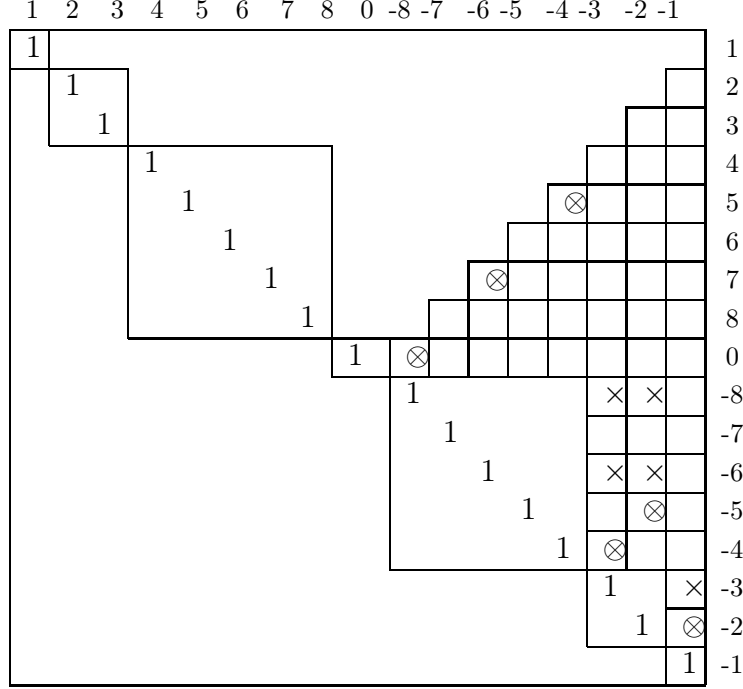


Diagram 3

We construct the formal matrix \mathbb{X} as follows. Let $\gamma \in M$. Let $\mathcal{E}(\gamma) = (-j, -i)$, where $i > 0$. Then the variable $x_{i,j}$ occupies the position (i, j) . The position $(-j, -i)$ is occupied by the variable $x_{i,j}$, where $G = Sp_{2n}(K)$ and $j < 0$, or by the variable $-x_{i,j}$ in the other cases. If $\mathcal{E}(\gamma) = (i, -i)$, $i > 0$, then the variable $x_{i,-i}$ stands in the position $(i, -i)$.

Assume that $\gamma \in M$ and $\mathcal{E}(\gamma) = (a, -b)$, $a \in \mathbb{Z}$, $b > 0$. Denote by S_γ the set of ξ in S such that $\mathcal{E}(\xi) = (i, j)$, $i \succ a$, and $j \prec -b$. Let $S_\gamma = \{\xi_1, \dots, \xi_k\}$, $\mathcal{E}(\xi_i) = (a_i, -b_i)$, where $a_i \in \mathbb{Z}$, $b_i > 0$ for any $1 \leq i \leq k$.

Let $b_{i_1}, b_{i_2}, \dots, b_{i_s}$ be numbers in the set $\{b_1, b_2, \dots, b_k\}$ that are greater than the number a .

Notation 11. Denote by M_γ the minor M_I^J of the matrix \mathbb{X} with ordered systems of rows I and columns J , where

$$I = \text{ord}\{b_1, \dots, b_k, b, a_{i_1}, \dots, a_{i_s}\}, \quad J = \text{ord}\{-a, -a_1, \dots, -a_k, -b_{i_1}, \dots, -b_{i_s}\}.$$

Denote by \overline{M}_γ the minor $M_{I'}^{J'}$ of the matrix \mathbb{X} , where

$$I' = \text{ord}\{a, a_1, \dots, a_k, b_{i_1}, \dots, b_{i_s}\}, \quad J' = \text{ord}\{-b_1, \dots, -b_k, -b, -a_{i_1}, \dots, -a_{i_s}\}.$$

Obviously, $M_\gamma = \pm \overline{M}_\gamma$.

Example 12. Let the group G and the parabolic subalgebra be as in Example 10. Let the root γ be the root $\varepsilon_6 + \varepsilon_7$; then $S_\gamma = \{\varepsilon_8\}$. We have $\mathcal{E}(\varepsilon_6 + \varepsilon_7) = (7, -6)$, $\mathcal{E}(\varepsilon_8) = (0, -8)$. Since $8 \succ 7$, we have $I = \{7, 8, 0\}$, $J = \{0, -8, -6\}$, whence we get

$$M_\gamma = \begin{vmatrix} x_{7,0} & x_{7,-8} & x_{7,-6} \\ x_{8,0} & 0 & -x_{6,-8} \\ 0 & -x_{8,0} & -x_{6,0} \end{vmatrix}.$$

Note that the statement $a_i \neq b_j$ is valid for any numbers $i \neq j$ such that $1 \leq i, j \leq k$. Indeed, we have $a_i > 0$, consequently, if $a_i = b_j$, then $\xi_i = \varepsilon_{b_i} + \varepsilon_{a_i}$ and $\xi_j = \varepsilon_{b_j} + \varepsilon_{a_j}$ if $a_j > 0$, and $\xi_j = \varepsilon_{b_j} - \varepsilon_{-a_j}$ if $a_j < 0$. Hence ξ_i and ξ_j of the base S are the comparable roots:

$$\xi_j - \xi_i = \varepsilon_{b_i} + \varepsilon_{a_i} - (\varepsilon_{a_i} \pm \varepsilon_{\pm a_j}) = \varepsilon_{b_i} \mp \varepsilon_{\pm a_j}.$$

We have a contradiction with the definition of a base. Thus the sets of rows I and columns J of the minor M_γ do not contain equal elements. Therefore M_γ and \overline{M}_γ are the square minors.

Suppose that a set A contains some numbers of the set

$$\{1, 2, \dots, n, 0, -n, \dots, -2, -1\}.$$

Denote

$$\min A = \{k \in A : k \preceq a \text{ for any } a \in A\},$$

$$\max A = \{k \in A : k \succeq a \text{ for any } a \in A\}.$$

The following lemma is needed in the sequel.

Lemma 13. Let $\xi \in S$, and rows I and columns J form the minor M_ξ . Let $\mathcal{E}(\xi) = (a, -b)$, $b > 0$.

1. Assume that the number $i \notin I$ such that $\min I \prec i \prec \max I$; then $i = a$.
2. For any number j such that $\min J \prec j \prec \max J$, we have $j \in J$.

PROOF.

1. We prove the first statement of the lemma. Let $i \notin I$, then, by the definition of the minor M_ξ , we conclude that there is no root $\gamma \in S$ such that $\mathcal{E}(\gamma) = (j, -i)$ and $i \preceq a$. Assume that $i \prec a$.

Consider roots $\gamma_j \in \Delta^+$ such that $\mathcal{E}(\gamma_j) = (j, -i)$. Denote by A the set of the numbers j such that the roots γ_j lie in M . As was said above, for any number $j \in A$ we have $\gamma_j \notin S$. By the definition of a base, we deduce that there exists a root $\xi_j \in S$ such that $\gamma_j - \xi_j \in \Delta^+$ for the given root γ_j . Thus any root $\alpha \in M$ such that $\mathcal{E}(\alpha) = (j, -k)$, where $j \in A$ and $-k \succ -i$, is comparable with the root ξ_j of the base S , i.e., any such root α does not lie in S . Further, since $\min I = b \prec i$, we have $-i \prec -b$. Since for any number $j \in J$ we have $j \prec \max J = -a$, it follows that $a \prec \min A$. By the assumption $i \prec a$, we have $i \prec \min A$. For any root $\alpha \in M$ such that $\mathcal{E}(\alpha) = (\min A, -i)$ we have $\min A \prec i$. We obtain a contradiction, and thus $i = a$.

2. Now we prove the second statement. Let j be a number such that $\min J \prec j \prec \max J$. Obviously, $\max J = -a$. Consider a root $\gamma \in M$ such that $\mathcal{E}(\gamma) = (-j, -b)$. Then

$$\gamma = \begin{cases} \varepsilon_b - \varepsilon_j & \text{if } j > 0; \\ \varepsilon_b + \varepsilon_{-j} & \text{if } j < 0. \end{cases}$$

Clearly, $\gamma \notin S$. Otherwise the root γ is comparable with the root $\xi \in S$. We obtain a contradiction with the definition of a base.

Let $\gamma = \varepsilon_b - \varepsilon_j$. From Lemma 3 it follows that there exists a root $\xi' \in S$ such that $\mathcal{E}(\xi') = (-j, -i)$ for some $-i \prec -b$. Since $j \prec \max J = -a$, we have $-j \succ a$ and $-i \prec -b$. From the last inequalities it follows that $\xi' \in S_\xi$. Hence, $j \in J$.

Similarly, if $\gamma = \varepsilon_b + \varepsilon_{-j}$, then from Lemma 3 it follows that there exists a root $\xi' \in S$ such that either $\mathcal{E}(\xi') = (-j, -i)$ for some $-i \prec -b$ or $\mathcal{E}(\xi') = (i, j)$ for some $i \succ -j$. In the first case, we obtain $\xi' \in S_\xi$ and $j \in J$. In the second case, we have $i \succ -j \succ a$ and $j \prec -a \prec -b$. Consequently, $\xi' \in S_\xi$. Since $-j \succ a$, by the definition of the minor M_ξ , we have $j \in J$. \square

Suppose a root $\varphi \in \Phi$ corresponds to an admissible pair $q = (\xi, \xi') \in Q$. We construct a polynomial

$$L_\varphi = \sum_{\substack{\alpha_1, \alpha_2 \in \Gamma_r \cup \{0\} \\ \alpha_1 + \alpha_2 = \alpha_q}} M_{\xi + \alpha_1} \overline{M}_{\alpha_2 + \xi'}. \quad (1)$$

Theorem 15 shows that the polynomials L_φ are N -invariants.

Example 14. We continue the calculations of Example 7-8. We construct some polynomials, using the roots of the expanded base.

$$M_{\varepsilon_6 - \varepsilon_7} = x_{6,7}, \quad M_{\varepsilon_5 - \varepsilon_8} = \begin{vmatrix} x_{5,7} & x_{5,8} \\ x_{6,7} & x_{6,8} \end{vmatrix}, \quad M_{\varepsilon_4 - \varepsilon_5} = x_{4,5}, \quad M_{\varepsilon_3 - \varepsilon_4} = x_{3,4},$$

$$M_{\varepsilon_2 - \varepsilon_6} = \begin{vmatrix} x_{2,4} & x_{2,5} & x_{2,6} \\ x_{3,4} & x_{3,5} & x_{3,6} \\ 0 & x_{4,5} & x_{4,6} \end{vmatrix},$$

$$L_{\varepsilon_6 + \varepsilon_7} = -x_{6,7}x_{6,-7} - x_{6,8}x_{6,-8},$$

$$L_{\varepsilon_6 + \varepsilon_8} = -x_{6,-8} \begin{vmatrix} x_{5,7} & x_{5,8} \\ x_{6,7} & x_{6,8} \end{vmatrix} - x_{6,8} \begin{vmatrix} x_{5,7} & x_{5,-8} \\ x_{6,7} & x_{6,-8} \end{vmatrix} - x_{6,7} \begin{vmatrix} x_{5,7} & x_{5,-7} \\ x_{6,7} & x_{6,-7} \end{vmatrix}.$$

$$L_{\varepsilon_5 + \varepsilon_8} = 2 \begin{vmatrix} x_{5,7} & x_{5,-8} \\ x_{6,7} & x_{6,-8} \end{vmatrix} \begin{vmatrix} x_{5,7} & x_{5,8} \\ x_{6,7} & x_{6,8} \end{vmatrix},$$

$$L_{\varepsilon_4 - \varepsilon_6} = -x_{4,6}x_{6,7} - x_{4,5}x_{5,7}.$$

By $E_{i,j}$ denote the standard elementary square matrix having unit in the (i,j) th entry and zeros in all other positions. To every root $\alpha \in \Delta^+$ corresponds a one-parameter subgroup $g_\alpha(t)$ of square $m \times m$ matrices, where $t \in K$:

$$g_\alpha(t) = \begin{cases} E + t(E_{i,j} - E_{-j,-i}) & \text{if } \alpha = \varepsilon_i - \varepsilon_j; \\ E + t(E_{i,-j} + E_{j,-i}) & \text{if } \alpha = \varepsilon_i + \varepsilon_j \text{ and } G = \mathrm{Sp}_{2n}(K); \\ E + t(E_{i,-j} - E_{j,-i}) & \text{if } \alpha = \varepsilon_i + \varepsilon_j \text{ and } G \neq \mathrm{Sp}_{2n}(K); \\ E + tE_{i,-i} & \text{if } \alpha = 2\varepsilon_i \text{ and } G = \mathrm{Sp}_{2n}(K); \\ E + t(E_{i,0} - E_{0,-i}) - \frac{t^2}{2}E_{i,-i} & \text{if } \alpha = \varepsilon_i \text{ and } G = \mathrm{O}_{2n+1}(K). \end{cases} \quad (2)$$

Now we prove the main statement of the paper.

Theorem 15. *For any parabolic subalgebra, the system of polynomials*

$$\{M_\xi, \xi \in S; L_\varphi, \varphi \in \Phi\}$$

is contained in $K[\mathfrak{m}]^N$ and is algebraically independent over K .

PROOF. It is well known that for any fixed ordering of positive roots, any element $g \in N$ can be written in the form

$$g = \prod_{\alpha \in \Delta^+} g_\alpha(t_\alpha),$$

where $t_\alpha \in K$ such that $\alpha \in \Delta^+$ are uniquely determined. Therefore it is sufficient to prove that for any $\xi \in S$ and $\varphi \in \Phi$ the polynomials M_ξ

and L_φ are invariants of the adjoint action of the one-parameter subgroups $g_\alpha(t)$ for any $\alpha \in \Delta^+$, $t \in K$. Note that the adjoint action by the element $E + tE_{i,j} \in U_m(K)$, $i < j$, reduces to the composition of two transformations: the row j multiplied by $-t$ is added to the row i ; the column i multiplied by t is added to the column j .

Let us show that the minor M_ξ , $\xi \in S$, is an invariant of the adjoint action of $g_\alpha(t)$. Let $\mathcal{E}(\xi) = (a, -b)$, $b > 0$. Assume that the rows I and the columns J form the minor M_ξ . We shell give two comments, the second one is a result of Lemma 13:

1. The elements (i, j) , where $\min I \preccurlyeq i \preccurlyeq \max I$ and $1 \preccurlyeq j \prec \min J$, and (l, k) , where $\max I \prec l \preccurlyeq n$ и $\min J \preccurlyeq k \preccurlyeq \max J$, of the matrix \mathbb{X} equal zero.
2. Suppose $\min I \preccurlyeq i \preccurlyeq \max I$; hence if $i \neq a$, then $i \in I$. If $\min J \preccurlyeq j \preccurlyeq \max J$, then $j \in J$.

If for any number i such that $\min I \prec i \prec \max I$ we have $i \in I$, then the above remarks imply that the minor M_ξ is an invariant.

Suppose $i \notin I$ and $\min I \prec i \prec \max I$; then, by Lemma 13, we have $i = a$. By the corollary of Lemma 3, we have $G = O_m(K)$. Let $I = \{a_1, \dots, a_k\}$. We prove that the minor M_ξ is an invariant. It is sufficient to show that M_ξ is an invariant of the adjoint action of $g_\alpha(t)$, where the root $\alpha \in \Delta^+$ is $\mathcal{E}(\alpha) = (-a, -a_t)$ for some t . Then

$$\text{Ad}_{g_\alpha} M_\xi = M_\xi \pm t M_{I'}^J,$$

where $I' = \text{ord}\{a_1, \dots, a_{t-1}, a, a_{t+1}, \dots, a_k\}$. Let us show that the minor $M_{I'}^J$ equals zero. Consider the sets

$$\tilde{I} = \text{ord}\{l \in I' : l \succ a\} \subset I' \text{ and } \tilde{J} = \text{ord}\{l : -l \in \tilde{I}\}.$$

Since $a = -\max J$, we get $\tilde{J} \subset J$. Hence $M_{I'}^J$ is the minor

$$M_{I'}^J = \begin{vmatrix} M_{I' \setminus \tilde{I}}^{J \setminus \tilde{J}} & M_{I' \setminus \tilde{I}}^{\tilde{J}} \\ 0 & M_{\tilde{I}}^{\tilde{J}} \end{vmatrix}.$$

Thus, $M_{I'}^J = M_{I' \setminus \tilde{I}}^{J \setminus \tilde{J}} \cdot M_{\tilde{I}}^{\tilde{J}}$. We show that the order of the minor $M_{\tilde{I}}^{\tilde{J}}$ is an odd number. If this is the case, then $M_{\tilde{I}}^{\tilde{J}}$ is an antisymmetric minor and the order of $M_{\tilde{I}}^{\tilde{J}}$ is an odd number, whence $M_{\tilde{I}}^{\tilde{J}}$ equals zero. Now let $\gamma \in S$, $\mathcal{E}(\gamma) = (c, -d)$, be a root such that $d \in \tilde{I}$. Then $d \succ a$; therefore by the

definition of M_ξ implies that $c \in I$. Since $a \prec d \prec c$, we have $c \in \tilde{I}$. Thus, any root $\gamma \in S$, whenever $\mathcal{E}(\gamma) = (c, -d)$ and $d \in \tilde{I}$, adds two rows and two columns to the sets I и J , respectively. But the number a is in the set \tilde{I} ; therefor $M_{\tilde{I}}^{\tilde{J}}$ is an antisymmetric minor of odd order. Hence $M_{\tilde{I}}^{\tilde{J}}$ equals zero. Consequently, M_ξ is an invariant for any $\xi \in S$.

Let us show that L_φ is an invariant for any $\varphi \in \Phi$. Let $q = (\xi, \xi')$ be an admissible pair and $\{\xi, \alpha_q, \xi'\}$ be a chain. Suppose that $\mathcal{E}(\xi) = (a, -b)$ and $\mathcal{E}(\xi') = (a', -b')$, where $b, b' > 0$, are such that $a \preccurlyeq a'$. Then $\varphi = \alpha_q + \xi'$.

1. Consider a case where $a' \succcurlyeq -R$. We have $a \neq a'$; otherwise the roots ξ and ξ' in S are comparable. Since $a' < 0$, we have $\xi' = \varepsilon_{b'} - \varepsilon_{-a'}$. The roots ξ, α_q, ξ' form a chain; therefor $\mathcal{E}(\alpha_q) = (-b, a')$, i.e., $\alpha_q = \varepsilon_{-a'} - \varepsilon_b$. Hence $\varphi = \varepsilon_{b'} - \varepsilon_b$ and $\mathcal{E}(\varphi) = (-b, -b')$.

Assume that we have the adjoint action of the subgroup $g_\alpha(t)$ on the polynomial L_φ , where $\mathcal{E}(\alpha) = (-j, -i)$, $i > 0$. If $-j \prec -b$ or $a' \preccurlyeq -i$, then the action of $g_\alpha(t)$ does not change L_φ . Therefore let $-b \preccurlyeq -j \prec -i \preccurlyeq a'$; then $j > 0$ and $\alpha = \varepsilon_i - \varepsilon_j$. Let the root α_q be the sum of two roots

$$\alpha_q = \gamma_1 + \alpha + \gamma_2, \text{ where } \gamma_1 = \varepsilon_j - \varepsilon_b, \gamma_2 = \varepsilon_{-a'} - \varepsilon_i.$$

Since $\alpha_q \in \Delta_{\mathfrak{t}}^+$, we have $\gamma_1, \gamma_2 \in \Delta_{\mathfrak{t}}^+$.

The following forms can be obtained by direct calculation.

$$\begin{aligned} \mathcal{E}(\xi + \gamma_1 + \alpha) &= (a, -i), & \mathcal{E}(\gamma_2 + \xi') &= (-i, -b'), \\ \mathcal{E}(\xi + \gamma_1) &= (a, -j), & \mathcal{E}(\alpha + \gamma_2 + \xi') &= (-j, -b'). \end{aligned}$$

Using (2), we have

$$\begin{aligned} \text{Ad}_{g_\alpha(t)} M_{\xi + \gamma_1 + \alpha} &= \frac{M_{\xi + \gamma_1 + \alpha} - t M_{\xi + \gamma_1}}{M_{\alpha + \gamma_2 + \xi'} + t M_{\gamma_2 + \xi'}}, \\ \text{Ad}_{g_\alpha(t)} \overline{M}_{\alpha + \gamma_2 + \xi'} &= \overline{M}_{\alpha + \gamma_2 + \xi'} + t \overline{M}_{\gamma_2 + \xi'}. \end{aligned} \quad (3)$$

Applying (3) to (1), we obtain

$$\begin{aligned} \text{Ad}_{g_\alpha(t)} L_\varphi - L_\varphi &= (M_{\xi + \gamma_1 + \alpha} - t M_{\xi + \gamma_1}) \overline{M}_{\gamma_2 + \xi'} + \\ &+ M_{\xi + \gamma_1} (\overline{M}_{\alpha + \gamma_2 + \xi'} + t \overline{M}_{\gamma_2 + \xi'}) - M_{\xi + \gamma_1 + \alpha} \overline{M}_{\gamma_2 + \xi'} - M_{\xi + \gamma_1} \overline{M}_{\alpha + \gamma_2 + \xi'} = 0, \end{aligned}$$

i.e., L_φ is an invariant.

2. Now suppose $a' \prec -R$. Obviously, since $\mathcal{E}(\alpha_q) = (-a, a')$, we have $a' < 0$. Therefore $\xi' = \varepsilon_{b'} - \varepsilon_{-a'}$. Now we write all variations of the roots ξ and α_q :

$$\xi = \begin{cases} \varepsilon_b - \varepsilon_{-a} & \text{if } a < 0; \\ \varepsilon_b + \varepsilon_a & \text{if } a > 0; \\ \varepsilon_b, & \text{if } a = 0; \end{cases} \quad \alpha_q = \begin{cases} \varepsilon_{-a'} + \varepsilon_{-a} & \text{if } a < 0; \\ \varepsilon_{-a'} - \varepsilon_a & \text{if } a > 0; \\ \varepsilon_{-a'} & \text{if } a = 0. \end{cases}$$

Consider the adjoint action of $g_\alpha(t)$. We prove that the polynomial L_φ is an invariant if $\alpha = \varepsilon_i - \varepsilon_j$ and $\alpha = \varepsilon_i$. The other cases $\alpha = \varepsilon_i + \varepsilon_j$ and $\alpha = 2\varepsilon_i$ are proved similarly.

- (a) Suppose that $\alpha = \varepsilon_i - \varepsilon_j$, $i < j$. Then $\mathcal{E}(\alpha) = (-j, -i)$. Obviously, $-a \prec a'$. The action of $g_\alpha(t)$ does not change the polynomial L_φ if $-i \succ a'$ and $-j \prec -a$. Assume that $-a \preccurlyeq -j \prec -i \prec a'$. First suppose $a > 0$. We write the root α_q as the sum of three roots, where α is one of these roots and the other roots are in Γ_r .

$$\alpha_q = \varepsilon_{-a'} - \varepsilon_a = \gamma_1 + \alpha + \gamma_2,$$

where $\gamma_1 = \varepsilon_j - \varepsilon_a$ and $\gamma_2 = \varepsilon_{-a'} - \varepsilon_i$. We have

$$\begin{aligned} \xi + \gamma_1 + \alpha &= \varepsilon_b + \varepsilon_i, & \gamma_2 + \xi' &= \varepsilon_{b'} - \varepsilon_i, \\ \xi + \gamma_1 &= \varepsilon_b + \varepsilon_j, & \alpha + \gamma_2 + \xi' &= \varepsilon_{b'} - \varepsilon_j. \end{aligned} \tag{4}$$

Let $a < 0$. As before, the root α_q is the sum of three roots:

$$\alpha_q = \varepsilon_{-a'} + \varepsilon_{-a} = \gamma_1 + \alpha + \gamma_2,$$

where $\gamma_1 = \varepsilon_{-a} + \varepsilon_j$ and $\gamma_2 = \varepsilon_{-a'} - \varepsilon_i$. Similarly, the relations (4) are verified by direct calculations. If $a = 0$, then $\alpha_q = \varepsilon_{-a'}$ is the sum of roots $\gamma_1 = \varepsilon_j$, α and $\gamma_2 = \varepsilon_{-a'} - \varepsilon_i$. As above, we get (4). Every case yields relations (3). Then

$$\text{Ad}_{g_\alpha(t)} L_\varphi - L_\varphi = 0.$$

- (b) Now suppose that $\alpha = \varepsilon_i$; then, by (2), we have

$$g_\alpha(t) = E + t(E_{i,0} - E_{0,-i}) - \frac{t^2}{2}E_{i,-i}.$$

The action of $g_\alpha(t)$ does not change the polynomial L_φ if $a' \preccurlyeq -i$ or $a \geq 0$. Hence let $a < 0$ and $a' \succ -i$. Then $\alpha_q = \varepsilon_{-a'} + \varepsilon_{-a}$. We

write the root α_q as the sum of two roots in $\Gamma_{\mathfrak{r}}$ such that a certain root of the sum equals $\alpha + \gamma$, $\gamma \in \Gamma_{\mathfrak{r}}$:

$$\alpha_q = \gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 = \gamma_5 + \gamma_6,$$

where

$$\begin{aligned} \gamma_1 &= \varepsilon_{-a} - \varepsilon_i, & \gamma_3 &= \varepsilon_{-a} + \varepsilon_i, & \gamma_5 &= \varepsilon_{-a}, \\ \gamma_2 &= \varepsilon_{-a'} + \varepsilon_i, & \gamma_4 &= \varepsilon_{-a'} - \varepsilon_i, & \gamma_6 &= \varepsilon_{-a'}. \end{aligned}$$

Then

$$\begin{aligned} \xi + \gamma_1 &= \varepsilon_b - \varepsilon_i, & \xi + \gamma_3 &= \varepsilon_b + \varepsilon_i, & \xi + \gamma_5 &= \varepsilon_b, \\ \gamma_2 + \xi' &= \varepsilon_{b'} + \varepsilon_i, & \gamma_4 + \xi' &= \varepsilon_{b'} - \varepsilon_i, & \gamma_6 + \xi' &= \varepsilon_{b'}. \end{aligned}$$

The adjoint action of the subgroup $g_\alpha(t)$ on the corresponding minors gives the following relations:

$$\begin{aligned} \text{Ad}_{g_\alpha(t)} \overline{M}_{\gamma_2+\xi'} &= \text{Ad}_{g_\alpha(t)} \overline{M}_{\varepsilon_{b'}+\varepsilon_i} = \overline{M}_{\varepsilon_{b'}+\varepsilon_i} - t \overline{M}_{\varepsilon_{b'}} - \frac{t^2}{2} \overline{M}_{\varepsilon_{b'}-\varepsilon_i}, \\ \text{Ad}_{g_\alpha(t)} M_{\xi+\gamma_3} &= \text{Ad}_{g_\alpha(t)} M_{\varepsilon_b+\varepsilon_i} = M_{\varepsilon_b+\varepsilon_i} - t M_{\varepsilon_b} - \frac{t^2}{2} M_{\varepsilon_b-\varepsilon_i}, \\ \text{Ad}_{g_\alpha(t)} M_{\xi+\gamma_5} &= \text{Ad}_{g_\alpha(t)} M_{\varepsilon_b} = M_{\varepsilon_b} + t M_{\varepsilon_b-\varepsilon_i}, \\ \text{Ad}_{g_\alpha(t)} \overline{M}_{\gamma_6+\xi'} &= \text{Ad}_{g_\alpha(t)} \overline{M}_{\varepsilon_{b'}} = \overline{M}_{\varepsilon_{b'}} + t \overline{M}_{\varepsilon_{b'}-\varepsilon_i}. \end{aligned}$$

Applying these expressions to (1), we get

$$\begin{aligned} \text{Ad}_{g_\alpha(t)} L_\varphi - L_\varphi &= M_{\varepsilon_b-\varepsilon_i} \left(\overline{M}_{\varepsilon_{b'}+\varepsilon_i} - t \overline{M}_{\varepsilon_{b'}} - \frac{t^2}{2} \overline{M}_{\varepsilon_{b'}-\varepsilon_i} \right) + \\ &\quad + \left(M_{\varepsilon_b+\varepsilon_i} - t M_{\varepsilon_b} - \frac{t^2}{2} M_{\varepsilon_b-\varepsilon_i} \right) \overline{M}_{\varepsilon_{b'}-\varepsilon_i} + \\ &\quad + \left(M_{\varepsilon_b} + t M_{\varepsilon_b-\varepsilon_i} \right) \left(\overline{M}_{\varepsilon_{b'}} + t \overline{M}_{\varepsilon_{b'}-\varepsilon_i} \right) - \\ &\quad - M_{\varepsilon_b-\varepsilon_i} \overline{M}_{\varepsilon_{b'}+\varepsilon_i} - M_{\varepsilon_b+\varepsilon_i} \overline{M}_{\varepsilon_{b'}-\varepsilon_i} - M_{\varepsilon_b} \overline{M}_{\varepsilon_{b'}} = 0. \end{aligned}$$

To complete the proof, it remains to show that for any $\xi \in S$ and $\varphi \in \Phi$, the polynomials M_ξ and L_φ are algebraically independent. Let

$$E_\gamma = \begin{cases} E_{i,j} - E_{-j,-i} & \text{if } \gamma = \varepsilon_i - \varepsilon_j; \\ E_{i,-j} - E_{j,-i} & \text{if } \gamma = \varepsilon_i + \varepsilon_j \text{ and } G \neq \text{Sp}_{2n}(K); \\ E_{i,-j} + E_{j,-i} & \text{if } \gamma = \varepsilon_i + \varepsilon_j \text{ and } G = \text{Sp}_{2n}(K); \\ E_{i,-i} & \text{if } \gamma = 2\varepsilon_i \text{ and } G = \text{Sp}_{2n}(K); \\ E_{i,0} - E_{0,-i} & \text{if } \gamma = \varepsilon_i \text{ and } G = \text{O}_{2n+1}(K). \end{cases}$$

Denote by \mathcal{Y} the subset in \mathfrak{m} of matrices of the form

$$\sum_{\xi \in S} c_\xi E_\xi + \sum_{\varphi \in \Phi} c'_\varphi E_\varphi.$$

Consider the restriction homomorphism $\pi(f) = f|_{\mathcal{Y}}$ of the algebra $K[\mathfrak{m}]$ to $K[\mathcal{Y}]$. The image $K[\mathfrak{m}]$ of the homomorphism is the polynomial algebra of $x_{i,j}$, where $(-j, -i) = \mathcal{E}(\gamma)$ for some root $\gamma \in S \cup \Phi$. Denote $x_{i,j} = x_\xi$ if $\mathcal{E}(\xi) = (-j, -i)$. We have the following images of the polynomials M_ξ and L_φ for any $\xi \in S$ and $\varphi \in \Phi$:

$$\pi(M_\xi) = \pm x_\xi \prod_{\gamma \in S_\xi} x_\gamma^{\delta_\gamma},$$

and if a root $\varphi \in \Phi$ corresponds to the admissible pair (ξ, ξ') , then

$$\pi(L_\varphi) = \pm x_\varphi \prod_{\gamma \in \{\xi\} \cup S_\xi \cup S_{\xi'}} x_\gamma^{\delta_\gamma},$$

where δ_γ equals 1 or 2. Since the system of $\pi(M_\xi)$ and $\pi(L_\varphi)$ is algebraically independent, where $\xi \in S$ and $\varphi \in \Phi$, then the system of M_ξ and L_φ is algebraically independent. \square

From Theorem 15 we obtain the following consequence.

Theorem 16. The dimension of N -orbit in \mathfrak{m} is no greater than the number

$$\dim \mathfrak{m} - |S| - |\Phi|.$$

References

- [1] V. L. Popov and E. B. Vinberg, "Invariant theory in: Progress in Science and technology, VINITI, Moscow (1989), pp. 137–309.
- [2] M. Goto and F. Grosshans. Semisimple Lie algebras, Lect. Notes in Pure and Appl. Math., Marcel Dekker, INC. (1978).
- [3] H. Kraft, Geometrische Methoden in der Invarianten-theorie, Braunschweig/Wiesbaden: Vieweg (1984).
- [4] A. N. Panov and V. V. Sevostaynova, "Regular N -orbits in the nilradical of a parabolic subalgebra in: Proceeding of the International Conference on Algebra and Number Theory (occasion of the 80th birthday of V. E. Voskresenskii), Samara (2007), pp. 152–161.
- [5] V. V. Sevostaynova, "The invariant field of the adjoint action of the unitriangular group in the nilradical of a parabolic subalgebra Journal of Math. Sciences, Vol. 171, No. 3 (2010), pp. 400–415.

- [6] V. V. Sevostaynova, "The algebra of invariants of the adjoint action of the unitriangular group in the nilradical of a parabolic subalgebra Vestnik SamGU, №2 (76) (2010), pp. 72–83.

DEPARTMENT OF MECHANICS AND MATHEMATICS, SAMARA STATE UNIVERSITY,
RUSSIA

E-mail address: victoria.sevostyanova@gmail.com